

ON THE UNIRATIONALITY OF 3-FOLD CONIC BUNDLES

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1. INTRODUCTION

A variety X is unirational if it is dominated by a rational variety. This was classically considered close to be rational and a long outstanding problem, after Lüroth and Castelnuovo, was to give examples of unirational non rational varieties. After decades of struggle three different approaches gave almost simultaneously the required examples, [CG] [IM] [AM]. Since then unirationality was considered old fashioned and it was gradually substituted by the more charming and powerful notion of rational connection. A variety is rationally connected if two general points can be joined by a rational curve. Rational connection is more suited for modern algebraic geometry and a great amount of results about rationally connected varieties come out of the theory of deformation of rational curves, see [Ko]. Clearly this posed a new quest, still open, for rationally connected non unirational varieties. This paper is not going in this direction, but, following the stream opened in [MM], aims to show that the two notions can cooperate. The main result is the following unirationality statement for 3-fold conic bundles, see definition 2.1.

Theorem 1.1. *Let T be a 3-fold and $\pi : T \rightarrow W$ a standard conic bundle outside a codimension 2. Let $\Delta \subset W$ be the discriminant curve of π . Assume that there is a base point free pencil of rational curves Λ in W , with $\Lambda \cdot \Delta \leq 7$. Assume that there is a very ample linear system \mathcal{M} with $\mathcal{M} \cdot \Lambda = 1$. Then T is unirational.*

To put Theorem 1.1 in the right perspective recall that Iskovskikh conjectured, [Is], that, except for a special well known case, a 3-fold standard conic bundle $\pi : T \rightarrow W$ with connected discriminant curve Δ is rational if and only if, up to birational isomorphism, there is a base point free pencil of rational curves Λ with $\Lambda \cdot \Delta \leq 3$. It is quite easy to prove the sufficiency of this conjecture, [Is], while the necessity is very hard and Shokurov was able to prove it under the additional hypothesis that W is either the plane or a minimal ruled surface \mathbb{F}_e , [Sh]. The following is probably the most natural application of the previous theorem

Corollary 1.2. *Let $\pi : T \rightarrow W$ be a standard 3-fold conic bundle. Let $\Delta \subset W$ be the discriminant curve. Assume that one of the following is satisfied:*

- $W \cong \mathbb{F}_e$ and $\Delta \sim aC_0 + bF$, with $a \leq 7$,
- $W \cong \mathbb{P}^2$ and $\deg \Delta \leq 8$,
- $W \cong \mathbb{P}^2$, $\deg \Delta = 9$, and Δ is singular.

Then T is unirational.

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In particular Corollary 1.2, thanks to Shokurov criteria, [Sh], produces infinitely many families of unirational 3-folds that are not rational.

Corollary 1.3. *Let $\pi : T \rightarrow \mathbb{F}_e$ be a standard 3-fold conic bundle. Let $\Delta \subset W$ be the discriminant curve. Assume that $\Delta \sim aC_0 + bF$, with $3 < a \leq 7$, then T is unirational and not rational.*

Let me briefly explain the main points of the proof. The first step is to use Enriques criterion and [GHS] to reduce the unirationality of T to the rational connection of some subvariety in $\text{Rat}^n(S)$, where S is a standard conic bundle surface. Then this subvariety is proved to be birational to a subvariety of $\text{Rat}^n(\mathbb{P}^2)$ that is seen to be a linear space.

The paper is organized as follows. The first section describes the reduction of the unirationality statement to a statement about rational connection of subvarieties of $\text{Rat}^n(S)$. The second section allows to substitute subvarieties in $\text{Rat}^n(S)$ with subvarieties in $\text{Rat}^n(\mathbb{P}^2)$. The third section proves the theorem.

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2. NOTATION AND CONVENTIONS

I work over an algebraically closed field k of characteristic zero. Let me start with the following definitions.

Definition 2.1. A standard conic bundle of dimension n is a flat morphism $\pi : Z \rightarrow W$ with

- $n = \dim Z = \dim W + 1$
- Z and W smooth
- $-K_Z$ π -ample

For a standard conic bundle let $F \cong \mathbb{P}^1$ be the general fiber. I say that $\pi : Z \rightarrow W$ is a standard conic bundle outside a codimension a if there is a dense open subset $U \subseteq W$, with $\text{codim } U^c \geq a$, such that $\pi|_{\pi^{-1}(U)}$ is a standard conic bundle.

Definition 2.2. Let $\mathbb{F}_e = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-e))$ be the Segre–Hirzebruch surface. Fix a conic bundle structure $\pi : S \rightarrow \mathbb{P}^1$, with general fiber F :

- $C_0 \subset S$ is the only section with negative self intersection if $e > 0$ or a fixed section if $e = 0$,
- $F_p := \pi^{-1}(\pi(p))$ is the fiber through the point $p \in S$.

I am interested in studying rational curves on standard conic bundles of dimension 2. I refer to [Ko] for all the necessary definitions. To do this I adopt the following convention and definitions.

Convention 2.3. *Let $\pi : S \rightarrow \mathbb{P}^1$ be a standard conic bundle of dimension 2. In the following $X, Y \subset S$ are such that:*

- X, Y are disjoint zero dimensional reduced subschemes,
- $\sharp(\pi(X \cup Y)) = \sharp(X \cup Y)$,
- for some $e \geq 0$, there is a birational map $\mu : S/\mathbb{P}^1 \rightarrow \mathbb{F}_e/\mathbb{P}^1$ such that $\mu(X \cup Y) \subset \mathbb{F}_e \setminus C_0$, and μ^{-1} is an isomorphism in a neighborhood of X .

Definition 2.4. Let S be a standard conic bundle of dimension 2. I define the algebraic sets

$$\mathcal{R}^X(Y; a, d)_S := \overline{\{\text{irreducible rational curves in } |\mathcal{I}_{X^{2a} \cup Y^a}(-aK_S + dF)|\}} \subset \text{Rat}^n(S).$$

Remark 2.5. *Let me stress that by construction the general element of any irreducible component of $\mathcal{R}^X(Y; a, d)_S$ is an irreducible rational curve. The varieties $\mathcal{R}^X(Y; a, d)_S$ can also be interpreted as a locally closed subset in $|-aK_S + dF|$. In this way it is a subvariety of the Severi variety $V(-aK_S + dF)$, see [CH] for a modern account. I will switch between the two descriptions according to the convenience of the moment.*

The final ingredient are de Jonquières transformations. To introduce them let me recall the quasi projective variety Bir_d° .

Whenever $f_1, f_2, f_3 \in k[x, y, z]_d$ are not all zero, let us denote by $[f_1, f_2, f_3]$ the equivalence class of the triplets (f_1, f_2, f_3) with respect to the relation $(f_1, f_2, f_3) \sim (\lambda f_1, \lambda f_2, \lambda f_3)$, for $\lambda \in k^*$. Consider $[f_1, f_2, f_3]$ as an element of $\mathbb{P}^{3N-1=3d(d+3)/2+2}$, where the homogeneous coordinates are all coefficients of the three polynomials f_1, f_2, f_3 , up to multiplication by the same nonzero scalar for all of them. Setting

$$\omega: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \quad \omega([x, y, z]) = [f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)],$$

let us define, according to [BCM],

$$\text{Bir}_d^\circ = \{[f_1, f_2, f_3] \mid \omega \text{ is birational and } \gcd(f_1, f_2, f_3) = 1\} \subset \mathbb{P}^{3N-1}.$$

If $[f_1, f_2, f_3] \in \text{Bir}_d^\circ$, let us identify it with the birational map ω .

Definition 2.6. Let $\omega = [f_1, f_2, f_3] \in \text{Bir}_d^\circ$, $d \geq 2$. Setting B the linear span $\langle f_1, f_2, f_3 \rangle$ of the polynomials f_1, f_2, f_3 in k^N , the plane $\mathbb{P}(B) \subset \mathbb{P}^{N-1}$ is called the *homaloidal net* associated to ω and I denote it by \mathcal{L}_ω .

The general element of \mathcal{L}_ω defines an irreducible rational plane curve of degree d passing through some fixed points p_1, \dots, p_r in \mathbb{P}^2 , called *set-theoretic base points* of \mathcal{L}_ω , with certain multiplicities.

Definition 2.7. The map $\omega \in \text{Bir}_d^\circ$ is called a *de Jonquières transformation* if there exists a base point of \mathcal{L}_ω with multiplicity $d - 1$.

Remark 2.8. *The inverse of a de Jonquières transformation of degree d is again a de Jonquières transformation of degree d . Let $\omega: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a de Jonquières map. Then \mathcal{L}_ω has $2d - 2$ base points $\{q_1, \dots, q_{2d-2}\}$, eventually infinitely near, of multiplicity 1. The simple base points are the residual intersection of a general element in \mathcal{L}_ω with a fixed curve of degree $d - 1$ having p_0 of multiplicity $d - 2$. The latter curve is contracted by ω to a smooth point, the base point of multiplicity $(d - 1)$ of the inverse.*

Convention 2.9. Let $p_0 := [1, 0, 0] \in \mathbb{P}^2$, I will say that a zero dimensional reduced subset $Z \subset \mathbb{P}^2 \setminus \{p_0\}$ satisfies condition (\dagger) if

$$(\dagger) \quad \langle q_i, p_0 \rangle \not\supset q_j \text{ for any } i \neq j.$$

Definition 2.10. I say that a de Jonquières transformation $\omega: \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$, of degree d , is centered at A if

- A satisfies condition (\dagger)
- \mathcal{L}_ω has multiplicity $d - 1$ in p_0

- $|A| = 2d - 2$
- A is the set of simple base points of ω .

3. FROM UNIRATIONALITY TO RATIONAL CONNECTION

In this section the unirationality problem of Theorem 1.1 is translated into a statement on rational connection of subvarieties in $\text{Rat}^n(S)$, where S is a standard conic bundle of dimension 2.

Let T be a 3-fold and $\pi : T \rightarrow W$ a conic bundle outside a codimension 2. Let $\Delta \subset W$ be the discriminant curve, that is the curve that describes the singular fibers. Let Λ be a base point free pencil of rational curves on W and $f : W \rightarrow \mathbb{P}^1$ the morphism associated to Λ . Consider the composition $\psi := f \circ \pi$

$$\begin{array}{ccc} T & \xrightarrow{\pi} & W \xrightarrow{f} \mathbb{P}^1 \\ & \searrow \psi & \nearrow \end{array}$$

Let $x \in \mathbb{P}^1$ be a general point and S_x the fiber of ψ over x . Then S_x is a standard conic bundle with δ reducible fibers. Let S_η be the generic fiber and

$$\overline{S} := S_\eta \otimes_{k(x)} \overline{\text{Spec } k(x)}$$

the algebraic closure. Then there is an $e \geq 0$ such that \overline{S} is the blow up in δ distinct points $\{p_1, \dots, p_\delta\} \subset \mathbb{F}_e \setminus C_0$ of the surface \mathbb{F}_e . The curve $C_0 \subset \overline{S}$ has self intersection $C_0^2 = -e$. Let \tilde{C}_0 be a curve conjugate with C_0 over k , then $\tilde{C}_0^2 = -e$. In particular the surface S_x can be seen as the blow up of \mathbb{F}_e in δ points in $\mathbb{F}_e \setminus C_0$.

I want to define subsets X and Y on the family of standard conic bundles S_x , keep in mind Convention 2.3. Assume that there is a very ample linear system \mathcal{M} such that $\mathcal{M} \cdot \Lambda = 1$.

Let $m_i \in \mathcal{M}$ be a general element and $D_{m_i} = \pi^{-1}(m_i)$ the corresponding surface in T . Then any section $\Sigma_{m_i} \subset D_{m_i}$ defines a point in S_x . Fix $\{\Sigma_{m_1}, \dots, \Sigma_{m_s}\}$ general sections, and non negative integers a and d . Let

$$Z_x = \cup_1^m \Sigma_i \cap S_x$$

and consider the subvarieties

$$\mathcal{R}(\Sigma_1, \dots, \Sigma_m; a, d) := \{([C], x) \mid C \in \mathcal{R}^{Z_x}(\emptyset; a, d)_{S_x}\} \subset \text{Rat}^n(T/\mathbb{P}^1).$$

Let

$$(1) \quad \nu : \mathcal{R}(\Sigma_1, \dots, \Sigma_m; a, d) \rightarrow \mathbb{P}^1$$

be the structure map.

Remark 3.1. We may interpret the generic fiber of ν as $\mathcal{R}^{Z_\eta}(\emptyset; a, d)_{S_\eta}$, where Z_η is the set of generic points in $\cup_1^m \Sigma_{l_i}$.

The following proposition is the bridge I am looking for.

Proposition 3.2. Let $\pi : T \rightarrow W$ be a 3-fold conic bundle outside a codimension 2. Assume that there is a base point free pencil of rational curves Λ in W , and a very ample linear system \mathcal{M} with $\mathcal{M} \cdot \Lambda = 1$. Let

$$\mathcal{R}(\Sigma_{m_1}, \dots, \Sigma_{m_s}; a, d) \subset \text{Rat}^n(T/\mathbb{P}^1)$$

be as above and assume that there is a variety $R \subseteq \mathcal{R}(\Sigma_{m_1}, \dots, \Sigma_{m_s}; a, d)$ such that for a general x : $R_x := R \cap \nu^{-1}(x)$ is rationally connected and the general element in R_x is irreducible. Then T is unirational.

Proof. The proof is an easy consequence of Enriques criterion and the main result in [GHS]. By Enriques criterion T is unirational if and only if there is a rational surface $D \subset T$ intersecting the general fiber of π . I am assuming that R_x is rationally connected. Then by [GHS] there is a section, say Γ , of the morphism ν in equation (1), through the general point of R . Then Γ gives the desired rational surface. \square

Remark 3.3. *I want to stress a further consequence of Proposition 3.2. Usually to prove (uni)rationality one has to work on non algebraically closed field to prove (uni)rationality results on the generic fiber of a morphism. Thanks to [GHS] the unirationality problem I am interested in is reduced to study the rational connection of subvarieties in $\mathcal{R}^Z(\emptyset; a, d)_S$, for a standard conic bundle S of dimension 2 defined over the algebraically closed field k .*

The next step is to substitute the standard conic bundle S with \mathbb{P}^2 .

4. FROM CONIC BUNDLES TO \mathbb{P}^2

Let $\pi : S \rightarrow \mathbb{P}^1$ be a standard conic bundle of dimension two. In this section it is described a variety in $\text{Rat}^n(\mathbb{P}^2)$ that is birational to the variety $\mathcal{R}^Z(\emptyset; a, d)_S$.

Let $\mu : S \rightarrow \mathbb{F}_e$ be the blow down to some \mathbb{F}_e such that the indeterminacy point of μ^{-1} , say Y' , satisfies the convention 2.3

$$\begin{array}{ccc} S & \xrightarrow{\pi} & \mathbb{P}^1 \\ \mu \downarrow & \nearrow \pi & \\ \mathbb{F}_e & & \end{array}$$

With this notations I have.

Proposition 4.1. *Fix $X, Y \subset \mathbb{F}_e$ with $Y' \subseteq Y$. Let $X_S = \mu^{-1}(X)$ and $Y_S = \mu^{-1}(Y \setminus Y')$ assume that $\mathcal{R}^{X_S}(Y_S; a, d)_S$ is not empty and $\mathcal{R}^X(Y; a, d)_{\mathbb{F}_e}$ is irreducible, then $\mathcal{R}^X(Y; a, d)_{\mathbb{F}_e}$ is birational to $\mathcal{R}^{X_S}(Y_S; a, d)_S$.*

Proof. I have

$$-K_S = \mu^*(-K_{\mathbb{F}_e}) - \sum_{y_i \in Y'} E_{y_i},$$

with E_{y_i} the exceptional divisors over the point $y_i \in Y'$, and

$$(-aK_S + dF) \cdot E_{y_i} = a.$$

Let $[C] \in \mathcal{R}^{X_S}(Y_S; a, d)_S$ be a general point then C is irreducible and the above computation shows that $[\mu(C)] \in \mathcal{R}^X(Y; a, d)_{\mathbb{F}_e}$. This produces an injective map

$$\Theta : \mathcal{R}^{X_S}(Y_S; a, d)_S \dashrightarrow \mathcal{R}^X(Y; a, d)_{\mathbb{F}_e},$$

given by $\Theta([C]) = [\mu(C)]$. By hypothesis $\mathcal{R}^X(Y; a, d)_{\mathbb{F}_e}$ is irreducible and the general element in $\Theta(\mathcal{R}^{X_S}(Y_S; a, d)_S)$ has multiplicity a along Y' . Hence the general element in $\mathcal{R}^X(Y; a, d)_{\mathbb{F}_e}$ has multiplicity a along Y' . This shows that the map

$$\Psi : \mathcal{R}^X(Y; a, d)_{\mathbb{F}_e} \dashrightarrow \mathcal{R}^{X_S}(Y_S; a, d)_S,$$

given by $\Psi([\Gamma]) = [\mu_*^{-1}(\Gamma)]$ is defined on a dense open set and proves the claim. \square

Proposition 4.1 allows me to substitute $\mathcal{R}^Z(\emptyset; a, d)_S$ with $\mathcal{R}^X(Y'; a, d)_{\mathbb{F}_e}$. The next step is to go to \mathbb{P}^2 . For this consider the following construction.

Construction 4.2. Let $X := \{x_1, \dots, x_{|e-1|+2k}\} \subset \mathbb{F}_e$. Fix a subset $X' \subset X$ of $|e-1|$ points. Let $\varphi : \mathbb{F}_e \dashrightarrow \mathbb{F}_1$ be the rational map obtained by $|e-1|$ elementary transformations centered in X' , and $\epsilon : \mathbb{F}_1 \rightarrow \mathbb{P}^2$ the contraction of the unique (-1) -curve to the point $p_0 \in \mathbb{P}^2$. Let $f : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the de Jonquières transformation of degree $k+1$ centered in $(\epsilon \circ \varphi)(X \setminus X')$, see Definition 2.10.

Definition 4.3. I call $\chi^X := f \circ \epsilon \circ \varphi$ the composition.

The map f factors through the blow up ϵ , therefore χ^X is independent of the choice of the subset $X' \subset X$.

Remark 4.4. Let $Y, X \subset \mathbb{F}_e$ be a subsets with $|X| = |e-1| + 2k$. Then the map χ^X is composed with a de Jonquières map of degree k . In particular $(\epsilon \circ \varphi)(Y)$ satisfies condition (\dagger) and for X general the points $\chi^X(Y)$ impose independent conditions on curves of degree k .

An elementary transformation on \mathbb{F}_e allows to switch to $\mathbb{F}_{|e-1|}$.

Lemma 4.5. Fix two subsets $X, Y \subset \mathbb{F}_e$, keep in mind Convention 2.3. Let $\Phi_p : \mathbb{F}_e \dashrightarrow \mathbb{F}_{|e-1|}$ be the elementary transformation centered at $p \in X$. Define $Y_p := \Phi_p(Y)$ and $X_p = \Phi_p(X \setminus \{p\})$. If $\mathcal{R}^{X_p}(Y_p; a, d-a)_{\mathbb{F}_{|e-1|}}$ is irreducible and $\mathcal{R}^X(Y; a, d)_{\mathbb{F}_e}$ is not empty then $\mathcal{R}^X(Y; a, d)_{\mathbb{F}_e}$ is birational to $\mathcal{R}^{X_p}(Y_p; a, d-a)_{\mathbb{F}_{|e-1|}}$.

Proof. Let $[\Gamma] \in \mathcal{R}^X(Y; a, d)_{\mathbb{F}_e}$ be any element, then $\Phi_p(\Gamma) \cdot F = 2a$ and

$$\Phi_p(\Gamma) \cdot C_0 = \Gamma \cdot C_0 = d - ae + 2a = (d - a) - (|e-1|)a + 2a.$$

This yields $[\Phi_p(\Gamma)] \sim -aK_{\mathbb{F}_{|e-1|}} + (d-a)F$. This produces an injective map

$$\Theta : \mathcal{R}^X(Y; a, d)_{\mathbb{F}_e} \dashrightarrow \mathcal{R}^{X_p}(Y_p; a, d-a)_{\mathbb{F}_{|e-1|}},$$

given by $\Theta([\Gamma]) = [\Phi_p(\Gamma)]$. To conclude observe that the general curve in $\mathcal{R}^X(Y; a, d)_{\mathbb{F}_e}$ is irreducible, therefore $\Theta([\Gamma])$ does not contain $\Phi_p(F_p)$. Hence the general curve in $\Theta(\mathcal{R}^X(Y; a, d)_{\mathbb{F}_e})$ does not contain $\Phi_p(F_p)$. Since $\mathcal{R}^{X_p}(Y_p; a, d-a)_{\mathbb{F}_{|e-1|}}$ is irreducible then the map

$$\Psi : \mathcal{R}^{X_p}(Y_p; a, d-a)_{\mathbb{F}_{|e-1|}} \dashrightarrow \mathcal{R}^X(Y; a, d)_{\mathbb{F}_e},$$

given by $\Psi([\Gamma]) = [(\Phi_p^{-1})_*(\Gamma)]$ is well defined on a dense open subset and proves the claim. \square

It is time to go to \mathbb{P}^2 .

Definition 4.6. Let $A \subset \mathbb{P}^2 \setminus \{p_0\}$ be a reduced 0 dimensional subscheme satisfying assumption (\dagger) . I define the algebraic sets

$$\mathcal{R}(A; a, d)_{\mathbb{P}^2} := \overline{\{\text{irreducible rational curves in } |\mathcal{I}_{p_0^{d-2a} \cup A^a}(d)| \subset \text{Rat}^n(\mathbb{P}^2)\}}.$$

Remark 4.7. When $d > 1$ the varieties $\mathcal{R}(A; a, d)_{\mathbb{P}^2}$ are linear sections of a Severi variety on \mathbb{F}_1 .

Lemma 4.8. *Let $\mu : \mathbb{F}_1 \rightarrow \mathbb{P}^2$ be the blow up of \mathbb{P}^2 in the point p_0 . Fix a subset $Y \subset \mathbb{F}_1$, remember Convention 2.3. If $\mathcal{R}(\mu(Y); a, d+3)_{\mathbb{P}^2}$ is irreducible and $\mathcal{R}^\emptyset(Y; a, d)_{\mathbb{F}_1}$ is not empty then $\mathcal{R}(\mu(Y); a, d+3)_{\mathbb{P}^2}$ is birational to $\mathcal{R}^\emptyset(Y; a, d)_{\mathbb{F}_1}$.*

Proof. As in the previous Lemma note that μ is an isomorphism in a neighborhood of Y . Let $[\Gamma] \in \mathcal{R}^\emptyset(Y; a, d)_{\mathbb{F}_1}$ be an irreducible curve. Then $\Gamma \cdot C_0 = d+3-2a$ and $\mu(\Gamma) \sim \mathcal{O}(d+3)$ has multiplicity $(d+3)-2a$ in p_0 . This yields

$$[\mu(\Gamma)] \in \mathcal{R}(\mu(Y); a, d+3)_{\mathbb{P}^2}.$$

Then the general element in $\mathcal{R}(\mu(Y); a, d+3)_{\mathbb{P}^2}$ is a curve with multiplicity $d+3-2a$ in the point p_0 , therefore the strict transform is in $\mathcal{R}^\emptyset(Y; a, d)_{\mathbb{F}_1}$. \square

These sum up to give the following proposition.

Proposition 4.9. *Let S be a standard conic bundle and $\mu : S \rightarrow \mathbb{F}_e$ the blow down to some \mathbb{F}_e such that the indeterminacy points of μ^{-1} , say Y' , satisfy convention 2.3. Let $X, Y \subset \mathbb{F}_e \setminus C_0$ be subsets of general points with $|X| = e-1+2k$, and $\chi^X : \mathbb{F}_e \dashrightarrow \mathbb{P}^2$ the associated birational modification in Construction 4.2. Assume that:*

- $Y \supseteq Y'$,
- $\mathcal{R}(\chi^X(Y); a, d+3-a(e-1+2k))_{\mathbb{P}^2}$ is irreducible and not empty,
- the general element in $\mathcal{R}(\chi^X(Y); a, d+3-a(e-1+2k))_{\mathbb{P}^2}$ has multiplicity $d+3-2a-a(e-1+2k)$ in p_0 , and does not contain the indeterminacy locus of $(\chi^X)^{-1}$,
- the general element in $\mathcal{R}(\chi^X(Y); d+3-a(e-1+2k))_{\mathbb{P}^2}$ has multiplicity a along $\chi^X(Y)$.

Then the variety $\mathcal{R}^{\mu^{-1}(X)}(\mu^{-1}(Y \setminus Y'); a, d)_S$ is birational to $\mathcal{R}(\chi^X(Y); a, d+3-a(e-1+2k))_{\mathbb{P}^2}$.

Proof. First observe that $\mathcal{R}^{\mu^{-1}(X)}(\mu^{-1}(Y \setminus Y'); a, d)_S$ is not empty. I am assuming that the general element in

$$\mathcal{R}(\chi^X(Y); a, d+3-a(e-1+2k))_{\mathbb{P}^2}$$

does not contain $(\chi^X)^{-1} \setminus \{p_0\}$, has multiplicity $d+3-2a-a(e-1+2k)$ in p_0 , and has multiplicity a in $\chi^X(Y)$. Let

$$[\Gamma] \in \mathcal{R}(\chi^X(Y); a, d+3-a(e-1+2k))_{\mathbb{P}^2}$$

be a general point, then $[(\chi^X)^{-1}(\Gamma)] \in \mathcal{R}^{\mu^{-1}(X)}(\mu^{-1}(Y \setminus Y'); a, d)_S$. Then I can apply backwards recursively Lemma 4.8, Lemma 4.5, and Lemma 4.1 to get the desired conclusion. \square

5. PROOF OF THE MAIN RESULT

The first step in the proof is to produce a rationally connected subvariety in some $\mathcal{R}(Y; a, d)_{\mathbb{P}^2}$.

Proposition 5.1. *Let $Y \subset \mathbb{P}^2 \setminus \{p_0\}$ be a set of points satisfying assumption (\dagger) , and f a general de Jonquières transformation of degree $d \geq 4$ with $\text{mult}_{p_0} \mathcal{L}_f = d-1$. Assume that $|Y| = 7$, then $\mathcal{R}(f(Y); 4, 11)_{\mathbb{P}^2} \cong \mathbb{P}^1$, moreover the general curve in $\mathcal{R}(f(Y); 4, 11)_{\mathbb{P}^2}$ is irreducible with 4-tuple points along $f(Y)$, and a 3-ple point in p_0 .*

Proof. A dimension count shows that $\mathcal{R}(Y; 4, 11)_{\mathbb{P}^2}$ is expected to be a linear space of dimension 1. Let $\{x_1, \dots, x_7\}$ be the points in Y , then I may assume that $f(Y)$ has neither three collinear points nor 6 points on a conic, see Remark 4.4. Let $\omega_1 : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the composition of the standard Cremona transformations centered in $\{x_1, x_2, x_3\}$ and $\{x_4, x_5, x_6\}$. Let $\{y_1, y_2, y_3\}$ and $\{z_1, z_2, z_3\}$ be the exceptional points of ω_1^{-1} . The general choice of f allows me to assume that $\{y_1, y_2, y_3, z_1, z_2, z_3, x_7\}$ have neither three collinear points nor 6 points on a conic. Let ω_1 be the composition of the standard Cremona transformations centered in $\{x_7, y_1, y_2\}$ and $\{y_3, p_0, z_1\}$. Let $\{w_1, w_2, w_3\}$ and $\{t_1, t_2, t_3\}$ be the exceptional points of ω_1^{-1} . Then again I may assume that $\{z_2, z_3, w_1, w_2, w_3, t_1, t_2, t_3\}$ are distinct points without three collinear points, 6 points on a conic, and they are not contained by a rational cubic curve. Let Λ be the pencil of quartic curves singular in $\{z_2, z_3, w_1\}$ and passing through $\{w_2, w_3, t_1, t_2\}$. Then a direct and straightforward computation shows that the strict transform linear system $(\omega_1 \circ \omega)^{-1}(\Lambda)$ is a pencil of irreducible curves of degree 11 with 4-tuple points in Y and a triple point in p_0 . The following table summarizes the computation.

deg	z_1	z_2	z_3	w_1	w_2	w_3	t_1	t_2
4	0	2	2	2	1	1	1	1
6	2	2	2	2	3	3	1	1
8	2	4	2	2	3	3	3	3
10	4	4	4	4	3	3	3	3
11	4	4	4	4	4	4	4	3

The columns indicate the degree of the curve and the corresponding multiplicities in the points. To pass from one row to the next apply the standard Cremona transformation centered in the bold points. \square

Remark 5.2. *It is reasonable to expect that the linear system in Proposition 5.1 is the best possible. In other words, in the notation of section 3, I expect that there are no subvarieties*

$$R \subset \mathcal{R}(\Sigma_{m_1}, \dots, \Sigma_{m_s}; a, d) \subset \text{Rat}^n(T/\mathbb{P}^1)$$

such that R_x is a linear space of positive dimension and the general element is irreducible as soon as $|Y| > 7$.

Proof of Theorem 1.1. In the notation of section 3 let S_x be a general fiber of the morphism $\psi : T \rightarrow \mathbb{P}^1$. Then S_x is a standard conic bundle of dimension 2 and I may assume that it is the blow up of \mathbb{F}_e along a subset Y' , of length $\delta \leq 7$, that satisfies the Convention 2.3. Let $\mu_x : S_x \rightarrow \mathbb{F}_e$ be any such map. Fix $s = e - 1 + 8$ general sections $\{\Sigma_1, \dots, \Sigma_s\}$ of the morphism ψ , and

$$Z_x = \cup_1^s \Sigma_i \cap S_x.$$

Fix $7 - \delta$ more general sections $\{\Sigma_{s+1}, \dots, \Sigma_{s+7-\delta}\}$ and let $Y_x = \cup_1^{7-\delta} \Sigma_{s+i} \cap S_x$.

By Construction 4.2 the map $\chi^{Z_x} \circ \mu_x$ is composed with a general de Jonquières transformation of degree 4. Then by Proposition 5.1 I have:

- i) $\mathcal{R}((\chi^{Z_x} \circ \mu_x)(Y' \cup Y_x); 4, 11)_{\mathbb{P}^2} \cong \mathbb{P}^1$,
- ii) the general curve in $\mathcal{R}((\chi^{Z_x} \circ \mu_x)(Y' \cup Y_x); 4, 11)_{\mathbb{P}^2}$ is irreducible with 4-tuple points along $(\chi^{Z_x} \circ \mu_x)(Y' \cup Y_x)$, and a 3-ple point in p_0 .

Then I may apply Proposition 4.9 to get

$$\mathcal{R}^{Z_x}(Y_x; 4, 36 + 4e)_{S_x} \cong \mathbb{P}^1.$$

Together with ii), keep also in mind Remark 2.5, this is enough to conclude by Proposition 3.2. \square

Proof of Corollary 1.2. Assume first that $W \cong \mathbb{F}_e$ with $f : W \rightarrow \mathbb{P}^1$ a conic bundle structure. The linear systems $\Lambda = f^*\mathcal{O}(1)$ and $\mathcal{M} \sim C_0 + (e + 1)f$ satisfies the assumption of Theorem 1.1 and allows to conclude.

If $W \cong \mathbb{P}^2$ fix a point $z \in \Delta$. Since π is a standard conic bundle then Δ has at most ordinary double points. Let $\mu : \mathbb{F}_1 \rightarrow \mathbb{P}^2$ be the blow up of z . It is known, [Sa, Proposition 2.4], that there is a standard conic bundle $\pi_1 : X_1 \rightarrow \mathbb{F}_1$ such that the following diagram is commutative

$$\begin{array}{ccc} X_1 & \xrightarrow{\epsilon} & X \\ \pi_1 \downarrow & & \downarrow \pi \\ \mathbb{F}_1 & \xrightarrow{\mu} & \mathbb{P}^2 \end{array}$$

Moreover the discriminant curve of π_1 is $\Delta_1 = \mu^{-1}(\Delta) \setminus C_0$, [Sa, Corollary 2.5]. That is $\Delta_1 \sim aC_0 + bF$ with $a = \deg \Delta - \text{mult}_z \Delta$. This is enough to conclude again by Theorem 1.1. \square

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